Killing some S-spaces by a coherent Suslin tree

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Motivation

Theorem (Kunen, Rowbottom, Solovay, etc). $\mathrm{MA}_{\aleph_1}$ *implies* $\mathcal{K}_2$*: Every ccc forcing has property K.*

Question (Todorčević). *Does* $\mathcal{K}_2$ *imply* $\mathrm{MA}_{\aleph_1}$*?*

Theorem (Todorčević). $\mathrm{PID} + \mathfrak{p} > \aleph_1$ *implies no* S*-spaces.*

Question (Todorčević). *Under* PID*, does no* S*-spaces imply* $\mathfrak{p} > \aleph_1$*?*

Definition (Todorčević). $\mathrm{PFA}(S)$ *is an axiom that there exists a coherent Suslin tree* S *such that the forcing axiom holds for every proper forcing which preserves* S *to be Suslin.*

Theorem (Farah). $\mathfrak{t} = \aleph_1$ *holds in the extension with a Suslin tree.*

Proof. Suppose that T is a Suslin tree, and take $\pi : T \to [\omega]^{\aleph_0}$ such that

$$s \leq_T t \to \pi(s) \supseteq^* \pi(t) \text{ and } s \perp_T t \to \pi(s) \cap \pi(t) \text{ finite.}$$

Then for a generic branch G through T, the set $\{\pi(s) : s \in G\}$ is a $\subseteq^*$-decreasing sequence which doesn't have a lower bound in $[\omega]^{\aleph_0}$. $\square$

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Definition (Todorčević). PFA(S) *is an axiom that there exists a coherent Suslin tree* S *such that the forcing axiom holds for every proper forcing which preserves* S *to be Suslin.*

Question (Todorčević). *Under* PFA(S), *does* S *force* $\mathcal{K}_2$*?*

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PFA(S) was introduced to combine many of the consequences of the two contradictory set theoretic axioms, the weak diamond principle, and PFA.

Theorem (Consequences from the weak $\diamondsuit$)**.** *A Suslin tree forces the following.*

(Farah) $\mathfrak{t} = \aleph_1$.

(Farah) *It doesn't hold that all $\aleph_1$-dense subsets of the reals are isomorphic.*

(Larson–Todorčević) *Every ladder system has an ununiformized coloring.*

(Larson–Todorčević) *There are no Q-sets.*

(Moore–Hrušák–Džamonja) $\diamondsuit(\mathbb{R}, \mathbb{R}, \neq)$ *holds.*

Theorem (Consequences from PFA)**.** *Under* PFA(S), S *forces the following.*

(Todorčević) $2^{\aleph_0} = \aleph_2 = \mathfrak{h} = \mathrm{add}(\mathcal{N})$.

(Farah) *The open graph dichotomy.*

(Todorčević) *The P-ideal dichotomy.*

(Todorčević) *There are no compact S-spaces.*

Theorem. *Under* PFA(S), S *forces the following.*

§**1.** *Every forcing with rectangle refining property has precaliber* $\aleph_1$.

§**2.** *There are no* ω_2*-Aronszajn trees.*

§**3.** *All Aronszajn trees are club-isomorphic.*

§**4.** *The weak club guessing and* $\mho$ *fail.*

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Recall. – An S-space is a hereditarily separable (HS) regular space which is not hereditarily Lindelöf, i.e. which has non-Lindelöf subspace.

– Every S-space has a right-separated subspace of order type ω_1.

– A right-separated regular space of order type ω_1 is an S-space iff it has no uncountable discrete subspace.

– Therefore, there are no S-spaces iff every right-separated regular space has an uncountable discrete subspace.

Theorem (Szentmiklóssy). $\mathsf{MA}_{\aleph_1}$ *implies no compact S-spaces.*

Theorem (Todorčević). *Under* PFA(S), S *forces no compact* S*-spaces.*

Recall. A coherent Suslin tree S consists of functions in $\omega^{<\omega_1}$ and closed under finite modifications. That is,

- for any s and t in S, $s \leq_S t$ iff $s \subseteq t$,
- S is closed under taking initial segments,
- for any s and t in S, $\{\alpha \in \min\{\mathrm{lv}(s), \mathrm{lv}(t)\}; s(\alpha) \neq t(\alpha)\}$ is finite, and
- for any $s \in S$ and $t \in \omega^{\mathrm{lv}(s)}$, if $\{\alpha \in \mathrm{lv}(s); s(\alpha) \neq t(\alpha)\}$ is finite, then $t \in S$.

For s and $t \in S$ with the same level, define

$$\begin{array}{cccc} \psi_{s,t} & \{u \in S; s \leq_S u\} & \to & \{u \in S; t \leq_S u\} \\ & \mathbin{\rotatebox[origin=c]{90}{$\in$}} & & \mathbin{\rotatebox[origin=c]{90}{$\in$}} \\ & u & \mapsto & t \cup (u \restriction [\mathrm{lv}(s), \mathrm{lv}(u))) \end{array}.$$

Note that $\psi_{s,t}$ is an isomorphism, and if s, t, u are nodes in S with the same level, then $\psi_{s,t}$, $\psi_{t,u}$ and $\psi_{s,u}$ commute.

Theorem (Todorčević). *Under* PFA(S), S *forces no compact* S*-spaces.*

In fact, he proved that under PFA(S), for any S-name for a non-Lindelöf space which is a subspace of some compact countably tight space, there exists a forcing notion which is proper and preserves S and adds an S-name for an uncountable discrete subspace.

Main Claim. *Let* $\dot{\tau}$ *be an* S*-name for a right-separated HS regular topology of order type* ω_1*, and suppose that* $\dot{\tau}$ *has the following property:*

- $\star$ *For any point* $\delta \in \omega_1$, S*-name* $\dot{U}$ *for an open nbhd of* δ, $\alpha \in \omega_1$, $t \in S_\alpha$ *and* $F \in [S_\alpha]^{<\aleph_0}$, *there exists an* S*-name* $\dot{U}'$ *for an open nbhd of* δ *such that* $t \Vdash_S$ "$\dot{U}' \subseteq \dot{U}$" *and for every* $s \in F$,

$$s \Vdash_S \text{“} \psi_{t,s}(\dot{U}') \text{ is open in } \dot{\tau} \text{”}.$$

Then there exists a forcing notion which is proper and preserves S *and adds an* S*-name for an uncountable discrete subspace.*

Main Claim. *Let $\dot{\tau}$ be an S-name for a right-separated HS regular topology of order type ω_1, and suppose that $\dot{\tau}$ has the following property:*

- $\star$ *For any point $\delta \in \omega_1$, S-name $\dot{U}$ for an open nbhd of δ, $\alpha \in \omega_1$, $t \in S_\alpha$ and $F \in [S_\alpha]^{<\aleph_0}$, there exists an S-name $\dot{U}'$ for an open nbhd of δ such that $t \Vdash_S$ " $\dot{U}' \subseteq \dot{U}$ " and for every $s \in F$,*

$$s \Vdash_S \text{`` } \psi_{t,s}(\dot{U}') \text{ is open in } \dot{\tau} \text{''}.$$

Then there exists a forcing notion which is proper and preserves S and adds an S-name for an uncountable discrete subspace.

In this property, for each $s \in F$, $s = \psi_{t,s}(t)$, and so it is true that

$$s \Vdash_S \text{`` } \psi_{t,s}(\dot{U}') \text{ is open in } \psi_{t,s}(\dot{\tau}) \text{''},$$

but it may happen that

$$s \nVdash_S \text{`` } \psi_{t,s}(\dot{U}') \text{ is open in } \dot{\tau} \text{''}.$$

However, for example, if $\dot{\tau}$ is an S-name for a topology generated by an open basis in the ground model, then $\star$ is satisfied.

Main Claim. *Let $\dot\tau$ be an S-name for a right-separated HS regular topology of order type ω_1, and suppose that $\dot\tau$ has the following property:*

- $\star$ *For any point $\delta \in \omega_1$, S-name $\dot U$ for an open nbhd of δ, $\alpha \in \omega_1$, $t \in S_\alpha$ and $F \in [S_\alpha]^{<\aleph_0}$, there exists an S-name $\dot U'$ for an open nbhd of δ such that $t \Vdash_S$ " $\dot U' \subseteq \dot U$ " and for every $s \in F$,*

$$s \Vdash_S \text{" } \psi_{t,s}(\dot U') \text{ is open in } \dot\tau \text{ "}.$$

Then there exists a forcing notion which is proper and preserves S and adds an S-name for an uncountable discrete subspace.

Corollary. *Under* PFA(S), *S forces that every right-separated regular topology of order type ω_1 whose topology generated by a basis in the ground model is not HS.*

Theorem (Rudin)**.** *If there exists a Suslin tree, so is an S-space.*

Theorem (Todorčević)**.** PFA *implies no S-spaces.*

Proof. Suppose that (ω_1, τ) is a right-separated HS regular space of order type ω_1.

Then for each point $\alpha \in \omega_1$, there exists an open nbhd U_α of α such that $\mathrm{cl}_\tau(U_\alpha) \cap [\alpha + 1, \omega_1) = \emptyset$. Define $U_{\omega_1} = \emptyset$.

We define a forcing notion $\mathbb{P}$ which consists of finite functions p such that

- dom(p) is a finite $\in$-chain of countable elementary submodels of $H(\aleph_2)$ with τ and $\langle U_\alpha; \alpha \in \omega_1 \rangle$, and ran$(p) \subseteq \omega_1 \cup \{\omega_1\}$, and
- for each $M, M' \in$ dom(p), if $M \in M'$ and $p(M) \neq \omega_1$, then $p(M) \in M' \setminus M$ and $p(M) \notin U_{p(M')}$,

ordered by extensions.

We show that it is proper, which finishes the proof.

Remember (ω_1, τ) is a right-separated HS regular space. $\alpha \in U_\alpha$ and $\mathrm{cl}_\tau(U_\alpha) \cap [\alpha+1, \omega_1) = \emptyset$.

- dom(p) is a finite $\in$-chain of countable elementary submodels of $H(\aleph_2)$ with τ and $\langle U_\alpha; \alpha \in \omega_1 \rangle$, and ran$(p) \subseteq \omega_1$, and
- for each $M, M' \in$ dom(p), if $M \in M'$ and $p(M) \neq \omega_1$, then $p(M) \in M' \setminus M$ and $p(M) \notin U_{p(M')}$.

Let $N \prec H(\theta)$ be ctbl with τ, $\langle U_\alpha; \alpha \in \omega_1 \rangle$ and $H(\aleph_2)$, and $p_0 \in \mathbb{P} \cap N$.

If there are no $\alpha \in \omega_1 \setminus N$ such that ran$(p_0) \cap U_\alpha \neq \emptyset$, then for every $q \leq_{\mathbb{P}} p_0$, ran$(q) \subseteq N$ holds, hence $p_0 \cup \left\{ \langle N \cap H(\aleph_2), \omega_1 \rangle \right\}$ is $(N, \mathbb{P})$-generic. This is not an interesting case.

So suppose that there are $\alpha \in \omega_1 \setminus N$ such that ran$(p_0) \cap U_\alpha = \emptyset$.

Remember (ω_1, τ) is a right-separated HS regular space. $\alpha \in U_\alpha$ and $\mathrm{cl}_\tau(U_\alpha) \cap [\alpha+1, \omega_1) = \emptyset$.

- $\mathrm{dom}(p)$ is a finite $\in$-chain of countable elementary submodels of $H(\aleph_2)$ with τ and $\langle U_\alpha; \alpha \in \omega_1 \rangle$, and $\mathrm{ran}(p) \subseteq \omega_1$, and
- for each $M, M' \in \mathrm{dom}(p)$, if $M \in M'$ and $p(M) \neq \omega_1$, then $p(M) \in M' \setminus M$ and $p(M) \notin U_{p(M')}$.

Let $N \prec H(\theta)$ be ctbl with τ, $\langle U_\alpha; \alpha \in \omega_1 \rangle$ and $H(\aleph_2)$, and $p_0 \in \mathbb{P} \cap N$.
We show that $p_1 := p_0 \cup \{\langle N \cap H(\aleph_2), \alpha \rangle\} \in \mathbb{P}$ is $(N, \mathbb{P})$-generic.

Let $\mathcal{D} \in N$ be dense open $\subseteq \mathbb{P}$ and $r \in \mathcal{D}$ with $r \leq_{\mathbb{P}} p_1$. Then

$$T := \left\{ \langle q(M) \rangle_{M \in \mathrm{dom}(q) \setminus \mathrm{dom}(r \cap N)} ; q \in \mathbb{P} \cap \mathcal{D} \text{ is an end-ext. of } r \cap N \ \& \ |q| = |r| \right\}$$

is in N and contains $\langle r(M) \rangle_{M \in \mathrm{dom}(r \setminus N)}$. Moreover, by thinking T as a tree which consists of all initial segments of its members, we can take a subtree $T' \subseteq T$ in N such that for every non-terminal member σ of T', $\{\beta \in \omega_1; \sigma^\frown \langle \beta \rangle \in T'\}$ is uncountable, and $\langle r(M) \rangle_{M \in \mathrm{dom}(r \setminus N)} \in T'$.

It is possible that there exists $\langle \beta_i; i < |r \setminus N| \rangle \in T' \cap N$ such that for any $i < |r \setminus N|$ and $M \in \mathrm{dom}(r \setminus N)$, $\beta_i \notin U_{r(M)}$.

Let $p_2 \in \mathbb{P} \cap N$ be a witness of $\langle \beta_i; i < |r \setminus N| \rangle \in T' \cap N$ and then $r \cup p_2 \in \mathbb{P}$, $\leq_{\mathbb{P}} p_2, r$.
□

Remember (ω_1, τ) is a right-separated HS regular space. $\alpha \in U_\alpha$ and $\mathrm{cl}_\tau(U_\alpha) \cap [\alpha+1, \omega_1) = \emptyset$.

- $\mathrm{dom}(p)$ is a finite $\in$-chain of countable elementary submodels of $H(\aleph_2)$ with τ and $\langle U_\alpha; \alpha \in \omega_1 \rangle$, and $\mathrm{ran}(p) \subseteq \omega_1$, and
- for each $M, M' \in \mathrm{dom}(p)$, if $M \in M'$ and $p(M) \neq \omega_1$, then $p(M) \in M' \setminus M$ and $p(M) \notin U_{p(M')}$.

The following is the crucial point of the proof that "there exists $\langle \beta_i; i < |r \setminus N| \rangle \in T' \cap N$ such that for any $i < |r \setminus N|$ and $M \in \mathrm{dom}(r \setminus N)$, $\beta_i \notin U_{r(M)}$".

Claim. *For any unctbl $X \subseteq \omega_1$ in N and $F \in [\omega_1 \setminus N]^{<\aleph_0}$, there exists $\beta \in X \cap N$ such that for every $\gamma \in F$, $\beta \notin U_\gamma$.*

Proof. There exists countable $Y \subseteq X$ in N such that $\mathrm{cl}_\tau(Y) = \mathrm{cl}_\tau(X)$, and take $\delta \in \mathrm{cl}_\tau(Y)$ such that for every $\gamma \in F$, $\gamma < \delta$.

Then there exists an open nbhd V of δ such that for every $\gamma \in F$, $\mathrm{cl}_\tau(U_\gamma) \cap V = \emptyset$.

Take $\beta \in Y \cap V$, then we are done. $\square$

Let S be a coherent Suslin tree and $\dot\tau$ an S-name for a regular topology on ω_1 such that $\Vdash_S$ " $(\omega_1, \dot\tau)$ is right-separated and HS".

For $\alpha \in \omega_1$, take an S-name $\dot U_\alpha$ s.t. $\Vdash_S$ " $\alpha \in \dot U_\alpha \in \dot\tau$ and $\mathrm{cl}_{\dot\tau}(\dot U_\alpha) \cap [\alpha+1, \omega_1) = \emptyset$".

$\mathbb{P}$ consists of finite functions p such that

- dom(p) is a finite $\in$-chain of c.e.s. of $H(\aleph_2)$ with S, $\dot\tau$ and $\left\langle \dot U_\alpha; \alpha \in \omega_1 \right\rangle$,
- for any $M \in$ dom(p), $p(M) = \left\langle t^p_M, \alpha^p_M \right\rangle \in (S \setminus M) \times (\omega_1 \setminus M)$,
- for any $M \in$ dom(p) and $\beta \in \omega_1 \cap M$, t^p_M decides whether $\beta \in \dot U_{\alpha^p_M}$ or not,
- for any $M, M' \in$ dom(p), if $M \in M'$, then $t^p_M, \alpha^p_M \in M'$, and
- for any $M, M' \in$ dom(p), if $t^p_M <_S t^p_{M'}$, then $t^p_{M'} \Vdash_S$ " $\alpha^p_M \notin \dot U_{\alpha^p_{M'}}$ ",

ordered by extensions.

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- for any $M \in \mathrm{dom}(p)$ and $\beta \in \omega_1 \cap M$, t^p_M decides whether $\beta \in \dot{U}_{\alpha^p_M}$ or not,
- for any $M, M' \in \mathrm{dom}(p)$, if $M \in M'$, then $t^p_M, \alpha^p_M \in M'$, and
- for any $M, M' \in \mathrm{dom}(p)$, if $t^p_M <_S t^p_{M'}$, then $t^p_{M'} \Vdash_S$ " $\alpha^p_M \notin \dot{U}_{\alpha^p_{M'}}$ ",

ordered by extensions.

If the following is true, then we can show that $\mathbb{P}$ is proper and preserves S.

- For any ctbl $N \prec H(\theta)$ with S, $\dot{\tau}$, $\left\langle \dot{U}_\alpha ; \alpha \in \omega_1 \right\rangle$ and $H(\aleph_2)$, $r \in \mathbb{P}$ with $r \cap N = \emptyset$, $u \in S$ with $\mathrm{lv}\left(t^r_M\right) \leq \mathrm{lv}(u)$ $\forall M \in \mathrm{dom}(r)$, and S-name $\dot{X} \in N$ for an unctbl $\subseteq \omega_1$, there exists $\beta \in \omega_1 \cap N$ such that $\forall M \in \mathrm{dom}(r)$, $t^r_M \leq_S u \rightarrow t^r_M \Vdash_S$ " $\beta \notin \dot{U}_{\alpha^r_M}$ ".

If the following is true, then we can show that $\mathbb{P}$ is proper and preserves S.

• For any ctbl $N \prec H(\theta)$ with S, $\dot{\tau}$, $\left\langle \dot{U}_\alpha; \alpha \in \omega_1 \right\rangle$ and $H(\aleph_2)$, $r \in \mathbb{P}$ with $r \cap N = \emptyset$, $u \in S$ with $\mathrm{lv}\left(t^r_M\right) \leq \mathrm{lv}(u)$ $\forall M \in \mathrm{dom}(r)$, and S-name $\dot{X} \in N$ for an unctbl $\subseteq \omega_1$, there exists $\beta \in \omega_1 \cap N$ such that $\forall M \in \mathrm{dom}(r)$, $t^r_M \leq_S u \to t^r_M \Vdash_S$ " $\beta \notin \dot{U}_{\alpha^r_M}$ ".

So, **Main Claim** follows from the next claim.

Claim. *Suppose that*

⋆ *For any point $\delta \in \omega_1$, S-name $\dot{U}$ for an open nbhd of δ, $\alpha \in \omega_1$, $t \in S_\alpha$ and $F \in [S_\alpha]^{<\aleph_0}$, there exists an S-name $\dot{U}'$ for an open nbhd of δ such that $t \Vdash_S$ " $\dot{U}' \subseteq \dot{U}$ " and for every $s \in F$,*

$$s \Vdash_S \text{`` } \psi_{t,s}(\dot{U}') \text{ is open in } \dot{\tau} \text{ ''}.$$

Then the above statement • holds.

• For any ctbl $N \prec H(\theta)$ with S, $\dot{\tau}$, $\left\langle \dot{U}_\alpha ; \alpha \in \omega_1 \right\rangle$ and $H(\aleph_2)$, $r \in \mathbb{P}$ with $r \cap N = \emptyset$, $u \in S$ with $\mathsf{lv}\left(t^r_M\right) \leq \mathsf{lv}(u)$ $\forall M \in \mathsf{dom}(r)$, and S-name $\dot{X} \in N$ for an unctbl $\subseteq \omega_1$, there exists $\beta \in \omega_1 \cap N$ such that $\forall M \in \mathsf{dom}(r)$, $t^r_M \leq_S u \rightarrow t^r_M \Vdash_S$ " $\beta \notin \dot{U}_{\alpha^r_M}$ ".

Claim. *Suppose that* $\star$ *For any point* $\delta \in \omega_1$, *S-name* $\dot{U}$ *for an open nbhd of* δ, $\alpha \in \omega_1$, $t \in S_\alpha$ *and* $F \in [S_\alpha]^{<\aleph_0}$, *there exists an S-name* $\dot{U}'$ *for an open nbhd of* δ *such that* $t \Vdash_S$ " $\dot{U}' \subseteq \dot{U}$ " *and for every* $s \in F$, $s \Vdash_S$ " $\psi_{t,s}(\dot{U}')$ *is open in* $\dot{\tau}$ ".

Then the above statement • holds.

Proof. There are $s \in S \cap N$ and ctbl $Y \in N$ such that $s \leq_S u \restriction (\omega_1 \cap N)$, $s \Vdash_S$ " $Y \subseteq \dot{X}$ and $\mathsf{cl}_{\dot{\tau}}(Y) = \mathsf{cl}_{\dot{\tau}}(\dot{X})$ ".

Let $\left\{M \in \mathsf{dom}(r); t^r_M \leq_S u\right\} = \left\{M_\zeta ; \zeta < k\right\}$ and take $w_\zeta \in S$, $\zeta < k$, and $\delta \in \omega_1$ such that $t^r_{M_\zeta} \leq_S w_\zeta$, all w_ζ is of same level, $\delta > \alpha^r_M$ $\forall M \in \mathsf{dom}(r)$, and

$$w_0 \Vdash_S \text{" } \delta \in \mathsf{cl}_{\dot{\tau}}(Y) \text{ "}.$$

By induction on $\zeta < k$, take an S-name $\dot{V}_\zeta$ such that

- $w_\zeta \Vdash_S$ " $\delta \in \dot{V}_\zeta \in \dot{\tau}$ and $\mathsf{cl}_{\dot{\tau}}(\dot{U}_{\alpha^r_{M_\zeta}}) \cap \dot{V}_\zeta = \emptyset$ ",
- for every $\zeta' \in k$, $w_{\zeta'} \Vdash_S$ " $\psi_{w_\zeta, w_{\zeta'}}(\dot{V}_\zeta)$ is open in $\dot{\tau}$ ", and
- $w_{\zeta+1} \Vdash_S$ " $\dot{V}_{\zeta+1} \subseteq \psi_{w_\zeta, w_{\zeta+1}}(\dot{V}_\zeta)$ ".

• For any ctbl $N \prec H(\theta)$ with S, $\dot{\tau}$, $\left\langle \dot{U}_\alpha; \alpha \in \omega_1 \right\rangle$ and $H(\aleph_2)$, $r \in \mathbb{P}$ with $r \cap N = \emptyset$, $u \in S$ with $\mathrm{lv}\left(t^r_M\right) \leq \mathrm{lv}(u)$ $\forall M \in \mathrm{dom}(r)$, and S-name $\dot{X} \in N$ for an unctbl $\subseteq \omega_1$, there exists $\beta \in \omega_1 \cap N$ such that $\forall M \in \mathrm{dom}(r)$, $t^r_M \leq_S u \rightarrow t^r_M \Vdash_S$ " $\beta \notin \dot{U}_{\alpha^r_M}$ ".

Claim. *Suppose that* $\star$ *.....*
Then the above statement $\bullet$ *holds.*

Proof. There are $s \in S \cap N$ and ctbl $Y \in N$ such that $s \leq_S u{\restriction}(\omega_1 \cap N)$, $s \Vdash_S$ " $Y \subseteq \dot{X}$ and $\mathrm{cl}_{\dot{\tau}}(Y) = \mathrm{cl}_{\dot{\tau}}(\dot{X})$ ".

Let $\left\{M \in \mathrm{dom}(r); t^r_M \leq_S u\right\} = \{M_\zeta; \zeta < k\}$ and take $w_\zeta \in S$, $\zeta < k$, and $\delta \in \omega_1$ such that $t^r_{M_\zeta} \leq_S w_\zeta$, all w_ζ is of same level, $\delta > \alpha^r_M$ $\forall M \in \mathrm{dom}(r)$, and $w_0 \Vdash_S$ " $\delta \in \mathrm{cl}_{\dot{\tau}}(Y)$ ".

By induction on $\zeta < k$, take an S-name $\dot{V}_\zeta$ such that

- $w_\zeta \Vdash_S$ " $\delta \in \dot{V}_\zeta \in \dot{\tau}$ and $\mathrm{cl}_{\dot{\tau}}(\dot{U}_{\alpha^r_{M_\zeta}}) \cap \dot{V}_\zeta = \emptyset$ ",
- for every $\zeta' \in k$, $w_{\zeta'} \Vdash_S$ " $\psi_{w_\zeta, w_{\zeta'}}(\dot{V}_\zeta)$ is open in $\dot{\tau}$ ", and
- $w_{\zeta+1} \Vdash_S$ " $\dot{V}_{\zeta+1} \subseteq \psi_{w_\zeta, w_{\zeta+1}}(\dot{V}_\zeta)$ ".

Take $\beta \in Y$ and $x \geq_S w_0$ s.t. $x \Vdash_S$ " $\beta \in \psi_{w_{k-1}, w_0}(\dot{V}_{k-1})$ ". Then for every $\zeta \in k$,

$$\psi_{w_0, w_\zeta}(x) \Vdash_S \text{" } \beta \in \psi_{w_{k-1}, w_\zeta}(\dot{V}_{k-1}) \subseteq \dot{V}_\zeta, \text{ hence } \beta \notin \dot{U}_{\alpha^r_{M_\zeta}} \text{ ".}$$

Remember

Let S be a coherent Suslin tree and $\dot{\tau}$ an S-name for a regular topology on ω_1 such that $\Vdash_S$ " $(\omega_1, \dot{\tau})$ is right-separated and HS".

For each $\alpha \in \omega_1$, take an S-name $\dot{U}_\alpha$ s.t. $\Vdash_S$ " $\alpha \in \dot{U}_\alpha \in \dot{\tau} \mathrm{cl}_{\dot{\tau}}(\dot{U}_\alpha) \cap [\alpha+1, \omega_1) = \emptyset$".

$\mathbb{P}$ consists of finite functions p such that

- dom(p) is a finite $\in$-chain of c.e.s. of $H(\aleph_2)$ with S, $\dot{\tau}$ and $\left\langle \dot{U}_\alpha; \alpha \in \omega_1 \right\rangle$,
- for any $M \in$ dom(p), $p(M) = \left\langle t^p_M, \alpha^p_M \right\rangle \in (S \setminus M) \times (\omega_1 \setminus M)$,
- for any $M \in$ dom(p) and $\beta \in \omega_1 \cap M$, t^p_M decides whether $\beta \in \dot{U}_{\alpha^p_M}$ or not,
- for any $M, M' \in$ dom(p), if $M \in M'$, then $t^p_M, \alpha^p_M \in M'$, and
- for any $M, M' \in$ dom(p), if $t^p_M <_S t^p_{M'}$, then $t^p_{M'} \Vdash_S$ " $\alpha^p_M \notin \dot{U}_{\alpha^p_{M'}}$",

ordered by extensions.

• For any ctbl $N \prec H(\theta)$ with S, $\dot{\tau}$, $\left\langle \dot{U}_\alpha ; \alpha \in \omega_1 \right\rangle$ and $H(\aleph_2)$, $r \in \mathbb{P}$ with $r \cap N = \emptyset$, $u \in S$ with $\mathrm{lv}\left(t^r_M\right) \leq \mathrm{lv}(u)$ $\forall M \in \mathrm{dom}(r)$, and S-name $\dot{X} \in N$ for an unctbl $\subseteq \omega_1$, there exists $\beta \in \omega_1 \cap N$ such that $\forall M \in \mathrm{dom}(r)$, $t^r_M \leq_S u \to t^r_M \Vdash_S$ " $\beta \notin \dot{U}_{\alpha^r_M}$ ".

Claim. *Suppose that* $\star$ *.....*
Then the above statement • *holds.*

Proof. There are $s \in S \cap N$ and ctbl $Y \in N$ such that $s \leq_S u{\restriction}(\omega_1 \cap N)$, $s \Vdash_S$ " $Y \subseteq \dot{X}$ and $\mathrm{cl}_{\dot{\tau}}(Y) = \mathrm{cl}_{\dot{\tau}}(\dot{X})$ ".

Let $\left\{M \in \mathrm{dom}(r); t^r_M \leq_S u\right\} = \{M_\zeta ; \zeta < k\}$ and take $w_\zeta \in S$, $\zeta < k$, and $\delta \in \omega_1$ such that $t^r_{M_\zeta} \leq_S w_\zeta$, all w_ζ is of same level, $\delta > \alpha^r_M$ $\forall M \in \mathrm{dom}(r)$, and $w_0 \Vdash_S$ " $\delta \in \mathrm{cl}_{\dot{\tau}}(Y)$ ".

By induction on $\zeta < k$, take an S-name $\dot{V}_\zeta$ such that

- $w_\zeta \Vdash_S$ " $\delta \in \dot{V}_\zeta \in \dot{\tau}$ and $\mathrm{cl}_{\dot{\tau}}(\dot{U}_{\alpha^r_{M_\zeta}}) \cap \dot{V}_\zeta = \emptyset$ ",
- for every $\zeta' \in k$, $w_{\zeta'} \Vdash_S$ " $\psi_{w_\zeta, w_{\zeta'}}(\dot{V}_\zeta)$ is open in $\dot{\tau}$ ", and
- $w_{\zeta+1} \Vdash_S$ " $\dot{V}_{\zeta+1} \subseteq \psi_{w_\zeta, w_{\zeta+1}}(\dot{V}_\zeta)$ ".

Take $\beta \in Y$ and $x \geq_S w_0$ s.t. $x \Vdash_S$ " $\beta \in \psi_{w_{k-1}, w_0}(\dot{V}_{k-1})$ ". Then for every $\zeta \in k$,

$$\psi_{w_0, w_\zeta}(x) \Vdash_S \text{" } \beta \in \psi_{w_{k-1}, w_\zeta}(\dot{V}_{k-1}) \subseteq \dot{V}_\zeta, \text{ hence } \beta \notin \dot{U}_{\alpha^r_{M_\zeta}} \text{ ".}$$

Therefore,

• For any ctbl $N \prec H(\theta)$ with S, $\dot\tau$, $\left\langle \dot U_\alpha; \alpha \in \omega_1 \right\rangle$ and $H(\aleph_2)$, $r \in \mathbb{P}$ with $r \cap N = \emptyset$, $u \in S$ with $\mathrm{lv}\left(t^r_M\right) \leq \mathrm{lv}(u)$ $\forall M \in \mathrm{dom}(r)$, and S-name $\dot X \in N$ for an unctbl $\subseteq \omega_1$, there exists $\beta \in \omega_1 \cap N$ such that $\forall M \in \mathrm{dom}(r)$, $t^r_M \leq_S u \to t^r_M \Vdash_S$ " $\beta \notin \dot U_{\alpha^r_M}$ ".

Claim. *Suppose that* $\star$ *.....*
Then the above statement • *holds.*

Proof. There are $s \in S \cap N$ and ctbl $Y \in N$ such that $s \leq_S u{\restriction}(\omega_1 \cap N)$,
$s \Vdash_S$ " $Y \subseteq \dot X$ and $\mathrm{cl}_{\dot\tau}(Y) = \mathrm{cl}_{\dot\tau}(\dot X)$ ".

Let $\left\{M \in \mathrm{dom}(r); t^r_M \leq_S u\right\} = \{M_\zeta; \zeta < k\}$ and take $w_\zeta \in S$, $\zeta < k$, and $\delta \in \omega_1$ such that $t^r_{M_\zeta} \leq_S w_\zeta$, all w_ζ is of same level, $\delta > \alpha^r_M$ $\forall M \in \mathrm{dom}(r)$, and $w_0 \Vdash_S$ " $\delta \in \mathrm{cl}_{\dot\tau}(Y)$ ".

By induction on $\zeta < k$, take an S-name $\dot V_\zeta$ such that

- $w_\zeta \Vdash_S$ " $\delta \in \dot V_\zeta \in \dot\tau$ and $\mathrm{cl}_{\dot\tau}(\dot U_{\alpha^r_{M_\zeta}}) \cap \dot V_\zeta = \emptyset$ ",
- for every $\zeta' \in k$, $w_{\zeta'} \Vdash_S$ " $\psi_{w_\zeta, w_{\zeta'}}(\dot V_\zeta)$ is open in $\dot\tau$ ", and
- $w_{\zeta+1} \Vdash_S$ " $\dot V_{\zeta+1} \subseteq \psi_{w_\zeta, w_{\zeta+1}}(\dot V_\zeta)$ ".

Take $\beta \in Y$ and $x \geq_S w_0$ s.t. $x \Vdash_S$ " $\beta \in \psi_{w_{k-1}, w_0}(\dot V_{k-1})$ ". Then for every $\zeta \in k$,

$$t^r_{M_\zeta} \Vdash_S \text{" } \beta \notin \dot U_{\alpha^r_{M_\zeta}} \text{ ".}$$

□